

No-go theorem and optimization of dynamical decoupling against noise with soft cutoff

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We study the performance of dynamical decoupling in suppressing decoherence caused by soft-cutoff Gaussian noise, using short-time expansion of the noise correlations and numerical optimization. For the noise correlations with odd power terms in the short-time expansion, there exists no dynamical decoupling scheme to eliminate the decoherence to arbitrary orders of the short time, regardless of the timing or pulse shaping of the control under the population conserving condition. We formulate the equations for optimizing pulse sequences for dynamical decoupling that minimizes decoherence up to the highest possible order of the short-time expansion. In particular, we show that the Carr-Purcell-Meiboom-Gill sequence is optimal in short-time limit for the noise correlations with a linear order term in the time expansion.

PACS numbers: 03.67.Pp, 03.65.Yz, 03.67.Lx, 82.56.Jn

I. INTRODUCTION

Quantum information processing [1] relies on the coherence of quantum systems. Unavoidable interactions between a quantum system and its environment (bath) introduce noise on the system and lead to error evolutions (decoherence) of the quantum system. Various methods have been proposed to combat the decoherence, including decoherence-free subspaces [2–4], error-correction codes [5, 6], and dynamical decoupling (DD) [7–9]. In particular, the DD scheme uses rapid unitary control pulses acting only on the systems to suppress the effects of the noise from the environments. DD has the advantages of suppressing decoherence without measurement, feedback, or redundant encoding [8]. DD originated from the seminal spin echo experiment [10], in which the effect of a static random magnetic field (inhomogeneous broadening) is canceled. And more complex DD pulse sequences, such as the Carr-Purcell-Meiboom-Gill (CPMG) sequence [11, 12], were designed to prolong the spin coherence time [13].

The early DD schemes only eliminate low-order errors, i.e., the errors of quantum evolutions up to some low order in the Magnus expansion. By unitary symmetrization procedure [7, 8], DD cancels the first order (i.e., leading order) errors. To eliminate errors to the second order in short time, mirror-symmetric arrangement of two DD sequences can be used [8]. The first explicit arbitrary M th order DD scheme, which suppresses errors to $O(T^{M+1})$ for short evolution time T , is the concatenated DD (CDD) [14, 15] proposed by Khodjasteh and Lidar. CDD sequences against pure dephasing were investigated for electron spin qubits in realistic solid-state systems with nuclear spins as baths [16–18]. Experiments [19–23] have tested the performance of CDD. CDD works for generic quantum systems coupled to a finite bath [24, 25]. However, since CDD uses recursively constructed pulse sequences to suppress decoherence, the number of pulses increases exponentially with the decoupling order. As pulse errors are inevitably introduced in each control pulse in ex-

periments, finding efficient DD schemes with fewer control pulses is desirable. A remarkable advance is the Uhrig DD (UDD) [26–29]. UDD is optimal in the short-time limit in the sense that it suppresses the pure dephasing of a qubit coupled to a finite bath to the M th order using only M qubit flips. The performance bounds for UDD against pure dephasing were established [30]. Shaped pulses [31, 32] of finite amplitude can be incorporated into UDD [31]. Many recent experimental studies [20, 23, 33–37] demonstrated the performance of UDD.

It is important to find efficient schemes to suppress general decoherence (including pure dephasing and population relaxation). Yang and Liu extended UDD to the suppression of population relaxation [29]. This inspired efficient ways to suppress the general decoherence of single qubits, including concatenation of UDD sequences (CUDD) [38] and a much more efficient one called quadratic DD (QDD) [25, 39–42] discovered by West *et al* [39]. Based on the proof in [29], Mukhtar *et al* generalized UDD to protect arbitrary multilevel systems with full prior knowledge of the initial states [43]. One can actually preserve the coherence of arbitrary multi-qubit systems by protecting a mutually orthogonal operation set (MOOS) [25]. By nesting UDD sequences for protecting the elements in the MOOS, the nested UDD (NUDD) [25, 42, 43] requires only a polynomially increasing number of pulses in the decoupling order. These universal DD schemes also work for analytically time-dependent baths [44].

The above-mentioned variations of UDD, however, rely on the finiteness of the baths, i.e., the existence of hard high-frequency cutoff in the noise spectra. Legitimate questions are: For quantum systems coupled to an infinite quantum bath or affected by soft-cutoff noise, can any DD be designed to eliminate the decoherence to arbitrary orders of precision in the short-time limit? And if yes, how can such DD be designed? Such questions have been previously addressed in some specific noise models. Comparing the efficiency of various DD sequences in suppressing pure dephasing of a qubit due to classical noise, Cywiński *et al* observed that if the noise spectrum cutoff is not reached, CPMG sequences [11, 12] actually performs better than CDD and UDD sequences [45]. Also, Pasini and Uhrig derived the equations for minimizing decoherence for power-law spectra, and found that the numer-

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ically optimized sequences resemble CPMG [46]. Chen and Liu proved that for telegraphlike noise the CPMG sequences are the most efficient scheme in protecting the qubit coherence in the short-time limit and the decoherence can be suppressed at most to the third order of short evolution time by DD [47]. These results suggest that for noise with soft cutoff in the spectrum, there are certain constraints on the efficacy of DD. However, no conclusion has been drawn on the performance of DD with arbitrary timing and pulse shaping for the general cases of soft-cutoff noise.

In this paper, we address the general question of the performance of DD against soft-cutoff noise, based on the short-time expansion of the noise correlations. The essence of time-expansion method is to avoid divergence in the integration of soft-cutoff noise in the frequency domain. We find that when the expansion of the noise correlation has odd-power terms in time, there exists no DD scheme to eliminate the decoherence to an arbitrary order of the short time, regardless of the timing and shaping of the control pulses under the population conserving condition. The odd-power terms in the time-expansion indicate that the noise has no hard high-frequency cutoff. Since for soft-cutoff noise the decoherence cannot be eliminated at a certain order of short time, we derive a set of equations to minimize the leading-order term in the short-time expansion while eliminating the lower orders. These equations are numerically solved for optimal solutions. In particular, for noise correlations with a linear order term in time, we prove that the CPMG sequences are optimal. For other noise correlations with odd-order terms, the minimum pulse interval of the optimized sequences is larger than in UDD sequences. This feature is important in realistic experiments when there is a minimum pulse switching time [48].

This paper is organized as follows: In Sec. II, we analyze the performance of DD via the short-time expansion of correlation functions, and we give the condition under which decoherence suppression to arbitrary order is impossible. The relation between the high-frequency cutoff and the short-time expansion of correlations is also discussed. In Sec. III, we derive the equations for sequence optimization and obtain optimal DD for noise correlations with odd-power expansion terms. Finally, the conclusions are drawn in Sec. IV.

II. NO-GO THEOREM ON DYNAMICAL DECOUPLING AGAINST NOISE WITH SOFT CUTOFF

We consider the pure-dephasing Hamiltonian for a single spin (qubit)

$$H = \frac{1}{2} \sigma_z [\omega_a + \beta(t)], \quad (1)$$

where $\sigma_z = |+\rangle\langle+| - |-\rangle\langle-|$ is the Pauli operator of the qubit, ω_a is the energy splitting of the qubit, and $\beta(t)$ describes random noise with average $\overline{\beta(t)} = 0$. Here the over bar denotes averaging over the noise realizations. We assume that the statistics of the noise fluctuations are Gaussian.

After a duration of free evolution time T , the noise induces between the two states $|\pm\rangle$ a random phase shift $\int_0^T \beta(t) dt$ that

destroys the quantum coherence. We can suppress the decoherence by DD control on the qubit. There is only one noise source $\beta(t)$ in the model Eq. (1), and to suppress the decoherence we need to protect a MOOS which consists of a Pauli operator σ_x (more generally $\sigma_x \cos \phi + \sigma_y \sin \phi$ with ϕ being real) [25]. We will prove later that DD can suppress the decoherence (i.e., the protection of the MOOS $\{\sigma_x\}$) only to a certain order of short evolution time for noise correlations that have odd-power expansion terms in time. We expect that the proof also applies to other quantum systems (e.g., multi-qubit systems) when the noise correlations have odd-power terms, since in those systems there are more noise sources and more system operators (e.g., a MOOS consisting of $L > 1$ Pauli operators) should be protected.

When we apply a sequence of instantaneous unitary operations σ_x at the moments T_1, T_2, \dots, T_N , the controlled evolution operator reads

$$U(T) = (\sigma_x)^N U(T_{N+1}, T_N) \sigma_x U(T_N, T_{N-1}) \cdots \times \sigma_x U(T_2, T_1) \sigma_x U(T_1, T_0), \quad (2)$$

where $T_0 = 0, T_{N+1} = T$, and the free evolution operator

$$U(T_{j+1}, T_j) = e^{-i \frac{\sigma_z}{2} \int_{T_j}^{T_{j+1}} [\omega_a + \beta(t)] dt}. \quad (3)$$

Note that when N is odd, we may apply an additional σ_x pulse at the end of the sequence for the identity evolution. Using

$$\sigma_x U(T_{j+1}, T_j) \sigma_x = e^{-i \frac{\sigma_z}{2} \int_{T_j}^{T_{j+1}} [-\omega_a - \beta(t)] dt}, \quad (4)$$

we write the evolution operator as

$$U(T) = e^{-i \frac{\sigma_z}{2} \int_0^T \omega_a F_\pi(t) dt} e^{-i \frac{\sigma_z}{2} \int_0^T \beta(t) F_\pi(t) dt}, \quad (5)$$

where we have defined the modulation function for instantaneous π -pulse sequences [26, 45]

$$F_\pi(t) = \begin{cases} (-1)^j & \text{for } t \in (T_j, T_{j+1}] \\ 0 & \text{for } t > T, \text{ or } t \leq 0 \end{cases}. \quad (6)$$

Under DD control, the qubit is flipped at different moments, and the random field $\beta(t)$ is modulated by the modulation function $F_\pi(t)$. For multilevel systems, the modulation functions resulting from instantaneous π -pulse sequences may have values not restricted to $\{\pm 1\}$ for $t \in (0, T]$. In Ref. [49], it is shown that for DD composed of specially engineered finite-duration pulses, the effective modulation functions can take values from $\{+1, -1, 0\}$ alternatively. We may also encounter effective modulation functions which are triangle wave functions during the time of system evolution [50]. For a more general analysis, we assume that the control conserves the populations and the phase modulation function $F_\pi(t)$ has a general form as

$$F(t) = \begin{cases} \text{a bounded function} & \text{for } t \in (0, T] \\ 0 & \text{for } t > T \text{ or } t \leq 0 \end{cases}. \quad (7)$$

At the moment T , the off-diagonal density matrix element of an ensemble is

$$\rho_{\uparrow\downarrow}(T) = \rho_{\uparrow\downarrow}(0) e^{-i \int_0^T \omega_a F(t) dt} \overline{e^{-i \int_0^T \beta(t) F(t) dt}}, \quad (8)$$

where $F(t) = F_\pi(t)$ for ideal instantaneous σ_x pulses on the qubit. The coherence is characterized by the ensemble-averaged phase factor

$$W(T) \equiv \overline{e^{-i \int_0^T \beta(t) F(t) dt}}. \quad (9)$$

For Gaussian noise, the ensemble-averaged phase factor $W(T)$ is determined by the two-point correlation function $\overline{\beta(t_1)\beta(t_2)}$ and $W(T)$ becomes [45, 51, 52]

$$W(T) = e^{-\chi(T)}, \quad (10)$$

where the phase correlation

$$\chi(T) = \frac{1}{2} \int_0^T dt_1 \int_0^T dt_2 \overline{\beta(t_1)\beta(t_2)} F(t_1) F(t_2), \quad (11)$$

can be written as the overlap between the noise power spectrum and a filter function determined by the Fourier transform of the modulation function [45]. The aim of DD design is to minimize $\chi(T)$.

We assume the noise is stationary, i.e., of time translation symmetry, $\overline{\beta(t_1)\beta(t_2)} = \overline{\beta(t_1 - t_2)\beta(0)}$. The noise correlation function has the symmetry $\overline{\beta(t)\beta(0)} = \overline{\beta(0)\beta(t)}$, since $\beta(t)$ and $\beta(0)$ commute [For quantum noise, this may not be true. But in Eq. (11), t_1 and t_2 can be exchanged without changing the integration. So the noise correlation can always be symmetrized]. These symmetries indicate that the noise correlation is an even function of time, i.e.,

$$\overline{\beta(t)\beta(0)} \equiv C_{\text{corr}}(t) = C_{\text{corr}}(|t|). \quad (12)$$

The noise correlation can be transformed from the noise power spectrum $S(\omega)$ as

$$\overline{\beta(t)\beta(0)} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S(\omega) e^{-i\omega t}. \quad (13)$$

To analyze the decoherence suppression in short-time limit, we expand the noise correlation function in power series

$$\overline{\beta(t)\beta(0)} \equiv \sum_{k=0}^{\infty} C_k |t|^k. \quad (14)$$

The existence of odd-order terms means that the correlation function is non-analytical, which indicates that the noise source cannot be a finite quantum bath. The noise with non-analytical correlation functions must come from the fluctuations of an infinite bath, since otherwise the unitary evolution of a finite quantum system will always lead to analytical correlation functions. For example, the noise correlation $\overline{\beta(t)\beta(0)} = e^{-|t|/t_c}$ has odd-order terms in the time expansion, and the noise has a Lorentz spectrum, which has a power-law decrease at high frequencies. We call this power-law decrease a soft high-frequency cutoff in the noise spectrum [46].

This kind of noise can be caused by Markovian (or instantaneous) collisions in the bath [53]. We can see from Eq. (13) that when the noise spectrum has hard high-frequency cutoff, the noise correlation function $\overline{\beta(t)\beta(0)}$ is analytic as a function of time and the expansion contains only even-order terms, $\overline{\beta(t)\beta(0)}_{\text{cutoff}} = \sum_{k=0}^{\infty} C_{2k} t^{2k}$.

Expanding the noise correlation in time has the advantage of avoiding the divergence of integration over frequency for noise with a soft high-frequency cutoff in the power spectrum. Therefore we can derive certain general results on the efficacy of general dynamical decoupling control against soft-cutoff noise. Also, the performance of DD in the short-time limit is directly derived from the time-domain expansion.

Using the expansion Eq. (14), we write

$$\chi(T) = \sum_{k=0}^{\infty} C_k \phi_k, \quad (15)$$

where the decoherence functions

$$\phi_k \equiv \int_0^T dt_1 \int_0^{t_1} dt_2 (t_1 - t_2)^k F(t_1) F(t_2). \quad (16)$$

Therefore if the modulation function $F(t)$ is designed to make

$$\{\phi_k = 0\}_{k=0}^M \equiv \{\phi_0 = 0, \dots, \phi_M = 0\}, \quad (17)$$

the decoherence in the short-time limit is eliminated to the M th order, i.e., $\chi(T) = O(T^{M+3})$. Note that $e^{-i \int_0^T \omega_a F(t) dt} = 1$ in Eq. (8) when $\phi_0 = 0$. The even-order decoherence functions ϕ_{2k} describe the decoherence due to the low-frequency noise, while the odd-order functions ϕ_{2k+1} arise from the soft high-frequency cutoff in the noise spectrum.

A. Analysis of the decoherence functions

Below we show that when the noise has soft high-frequency cutoff, it is not possible to suppress the decoherence to an arbitrary order, no matter how the DD modulation function $F(t)$ is designed.

The general filter function is defined as the Fourier transform of the general modulation function,

$$\tilde{F}(\omega) \equiv \int_{-\infty}^{\infty} F(t) e^{i\omega t} dt = \tilde{F}^*(-\omega). \quad (18)$$

We expand it in power series

$$\tilde{F}(\omega) = \sum_{m=1}^{\infty} \frac{(i\omega)^{m-1}}{m!} \Lambda_m, \quad (19a)$$

$$\Lambda_m \equiv m \int_0^T F(t) t^{m-1} dt. \quad (19b)$$

Note that $\Lambda_m = O(T^m)$. For a sequence of N instantaneous π pulses, $F(t)$ is given by Eq. (6) and Λ_m reads

$$\Lambda_m^{(\pi)} = m \int_0^T F_\pi(t) t^{m-1} dt = \sum_{j=0}^N (-1)^j (T_{j+1}^m - T_j^m). \quad (20)$$

As $F(t) = 0$ for $t \notin (0, T]$, we extend the bounds of integration for t to infinity and transform Eq. (16) to

$$\phi_k = \frac{\partial^k}{\partial(i\eta)^k} \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \int_{-\infty}^{\infty} dt_1 \int_0^{t_1} dt_2 \times \tilde{F}(\omega_1) \tilde{F}(\omega_2) e^{-i\omega_1 t_1} e^{-i\omega_2 t_2} e^{i\eta(t_1 - t_2)}, \quad (21)$$

where we set $\eta \rightarrow 0$ after differentiation. Integrations over t_2 , t_1 and ω_1 give

$$\phi_k = \frac{\partial^k}{\partial(i\eta)^k} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\tilde{F}(\omega)}{i(\omega + \eta)} [\tilde{F}(\eta) - \tilde{F}(-\omega)]. \quad (22)$$

Using

$$\left. \frac{\partial^r}{\partial(i\eta)^r} \tilde{F}(\eta) \right|_{\eta=0} = \frac{r!}{(r+1)!} \Lambda_{r+1}, \text{ for } r \geq 0, \quad (23)$$

and changing the summation index, we obtain

$$\phi_k = k! \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[\frac{|\tilde{F}(\omega)|^2}{(-i\omega)^{k+1}} - \sum_{r=1}^{k+1} \frac{\tilde{F}(\omega)}{(-i\omega)^{k-r+2}} \frac{\Lambda_r}{r!} \right]. \quad (24)$$

Note that the summation over r and the integration over frequency cannot be exchanged when the integration does not converge for each individual term. Using Eq. (19a), we have

$$\begin{aligned} \phi_k &= k! \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[\frac{|\tilde{F}(\omega)|^2}{(-i\omega)^{k+1}} - \sum_{r=1}^{k+1} \sum_{n=1}^{k-r+2} \frac{(-1)^{k-r}}{(i\omega)^{k-r+3-n}} \frac{\Lambda_r}{r!} \frac{\Lambda_n}{n!} \right. \\ &\quad \left. - \sum_{r=1}^{k+1} \sum_{n=k-r+3}^{\infty} \frac{(-1)^{k-r}}{(i\omega)^{k-r+3-n}} \frac{\Lambda_r}{r!} \frac{\Lambda_n}{n!} \right]. \end{aligned} \quad (25)$$

We simplify the last line by using Eq. (19b) and the equality

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_{n=r}^{\infty} \frac{(\pm i\omega)^{n-r}}{n!} t^n = \frac{1}{2} \frac{t^{r-1}}{(r-1)!}, \text{ for } r \geq 1, t \geq 0, \quad (26)$$

which is proved in Appendix A. We obtain

$$\begin{aligned} \phi_k &= k! \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[\frac{|\tilde{F}(\omega)|^2}{(-i\omega)^{k+1}} - \sum_{r=1}^{k+1} \sum_{n=1}^{k-r+2} \frac{(-1)^{k-r}}{(i\omega)^{k-r+3-n}} \frac{\Lambda_r}{r!} \frac{\Lambda_n}{n!} \right] \\ &\quad - \frac{k!}{2} \sum_{r=1}^{k+1} (-1)^{k-r} \frac{\Lambda_r}{r!} \frac{\Lambda_{k-r+2}}{(k-r+2)!}. \end{aligned} \quad (27)$$

1. Even-order decoherence functions ϕ_{2m}

We first consider the even-order decoherence functions ϕ_{2m} , which is caused by the noise with hard high-frequency cutoff. For even number $2m$, $\frac{|\tilde{F}(\omega)|^2}{(-i\omega)^{2m+1}}$ is an odd function and its integral vanishes. Eq. (27) gives

$$\begin{aligned} \phi_{2m} &= (2m)! \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[\sum_{r=1}^{2m+1} \sum_{n=1}^{2m-r+2} \frac{-(-1)^r}{(i\omega)^{2m-r+3-n}} \frac{\Lambda_r}{r!} \frac{\Lambda_n}{n!} \right] \\ &\quad - \frac{(2m)!}{2} \sum_{r=1}^{2m+1} (-1)^{2m-r} \frac{\Lambda_r}{r!} \frac{\Lambda_{2m-r+2}}{(2m-r+2)!}, \end{aligned} \quad (28)$$

which is decomposed as (with the changes of summation order and indices)

$$\begin{aligned} \phi_{2m} &= (2m)! \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[\sum_{r=1}^{2m+1} \sum_{n=1}^{2m-r+2} \frac{-(-1)^r}{2(i\omega)^{2m-r+3-n}} \frac{\Lambda_r}{r!} \frac{\Lambda_n}{n!} \right. \\ &\quad \left. + \sum_{r=1}^{2m+1} \sum_{n=1}^{2m-r+2} \frac{-(-1)^n}{2(i\omega)^{2m-r+3-n}} \frac{\Lambda_r}{r!} \frac{\Lambda_n}{n!} \right] \\ &\quad - \frac{(2m)!}{2} \sum_{r=1}^{2m+1} (-1)^{2m-r} \frac{\Lambda_r}{r!} \frac{\Lambda_{2m-r+2}}{(2m-r+2)!}. \end{aligned} \quad (29)$$

The integrals of the terms with odd-power of ω vanish. And for even functions of ω , the sum $n+r$ is an odd number, so $(-1)^r + (-1)^n = 0$. Thus the integral in Eq. (29) vanishes. With the change of summation index of the last line in Eq. (29), we write

$$\phi_{2m} = (2m)! \left[\frac{(-1)^m}{2} \left(\frac{\Lambda_{m+1}}{(m+1)!} \right)^2 - \sum_{r=1}^m (-1)^r \frac{\Lambda_r}{r!} \frac{\Lambda_{2m-r+2}}{(2m-r+2)!} \right]. \quad (30)$$

From Eq. (30), we find that the following two sets of equations are equivalent

$$\{\phi_{2m} = 0\}_{m=0}^{M-1} \Leftrightarrow \{\Lambda_m = 0\}_{m=1}^M. \quad (31)$$

For instantaneous π -pulse sequences, the optimal solution of the equation set $\{\Lambda_m = \Lambda_m^{(\pi)} = 0\}_{m=1}^M$ is

$$T_j^{\text{UDD}} = T \sin^2 \left[\frac{\pi j}{2N+2} \right], \quad (j = 1, 2, \dots, N), \quad (32)$$

which is the timing of UDD sequences [26]. Note that the decoherence functions ϕ_{2m} describe the noise with high-frequency cutoff in spectrum. Our result is consistent with Refs. [28, 29]. The conditions Eq. (31) are more general than the one that leads to UDD, and Eq. (31) may lead to more general designs of optimal DD.

2. Leading odd-order decoherence function ϕ_{2M-1}

For the noise with soft high-frequency cutoff, the noise correlation function $\overline{\beta(t)\beta(0)}$ is not analytic. Its expansion contains odd-order terms $C_{2m+1}\phi_{2m+1} \neq 0$. Suppose the DD scheme has eliminated the leading order errors by making $\{\phi_{2m} = 0\}_{m=0}^{M-1}$, i.e., Eq. (31). We want to study the leading odd-order decoherence functions ϕ_{2M-1} due to the soft high-frequency cutoff in the noise spectrum.

The odd-order decoherence function ϕ_{2M-1} of Eq. (27) reads

$$\begin{aligned} \phi_{2M-1} &= \frac{(2M-1)!}{2} \sum_{r=1}^{2M} (-1)^r \frac{\Lambda_r}{r!} \frac{\Lambda_{2M+1-r}}{(2M+1-r)!} \\ &\quad + \frac{(2M-1)!}{2\pi} \int_{-\infty}^{\infty} d\omega \left[\frac{|\tilde{F}(\omega)|^2}{(-i\omega)^{2M}} \right. \\ &\quad \left. + \sum_{r=1}^{2M} \sum_{n=1}^{2M-r+1} \frac{(-1)^r}{(i\omega)^{2M-r+2-n}} \frac{\Lambda_r}{r!} \frac{\Lambda_n}{n!} \right]. \end{aligned} \quad (33)$$

The condition Eq. (31) gives

$$\phi_{2M-1} = (2M-1)! \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[\frac{|\tilde{F}(\omega)|^2}{(-i\omega)^{2M}} + \sum_{r=M+1}^{2M} \sum_{n=M+1}^{2M+1-r} \frac{(-1)^r}{(i\omega)^{2M-r+2-n}} \frac{\Lambda_r}{r!} \frac{\Lambda_n}{n!} \right]. \quad (34)$$

Notice in the summation $2M+1-r < M+1$. We obtain

$$\phi_{2M-1} = (-1)^M (2M-1)! \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{|\tilde{F}(\omega)|^2}{\omega^{2M}}. \quad (35)$$

In Eq. (35), the integrand $|\tilde{F}(\omega)|^2/\omega^{2M} \geq 0$ and it cannot vanish for all ω from $-\infty$ to ∞ . Thus we have the following theorem.

Theorem 1. *For the noise correlation $\overline{\beta(t)\beta(0)} = \sum_{k=0}^{\infty} C_k |t|^k$ with $\{C_{2m} \neq 0\}_{m=0}^{M-1}$ and $C_{2M-1} \neq 0$, there is no DD scheme that can induce a filter function $F(t)$ to eliminate the errors up to $O(T^{2M+1})$ for short evolution time T . That is, there is no DD scheme satisfying $\{\phi_{2m} = 0\}_{m=0}^{M-1}$ and $\phi_{2M-1} = 0$.*

This theorem holds for arbitrary non-zero modulation function $F(t)$, and it shows that one cannot suppress the decoherence to arbitrary order when the noise correlation is not analytic. For example, for the noise with the correlation function $e^{-\gamma|t|}$, one cannot simultaneously eliminate the two leading decoherence terms $C_0\phi_0$ and $C_1\phi_1$, and the error induced by the noise is at least of the order $O(T^3)$.

The result in Ref. [47] that DD can suppress decoherence at most to the third order of short evolution time for telegraphlike noise is general for noise with arbitrary statistics. In deriving Theorem 1, we have made the assumption that the statistics of the noise are Gaussian and the decoherence is determined by the two-point noise correlation. For non-Gaussian noise, there exists the decoherence effect from higher-order noise correlations.

B. Noise correlation expansion and high-frequency cutoff

Here we discuss further on the relation between the noise correlation expansion and high-frequency cutoff. Let us consider the correlation functions of the expansion form

$$\overline{\beta(t)\beta(0)} = C_{\text{corr}}(t) = \sum_{k=0}^{\infty} C_k |t|^k, \quad (36)$$

where the expansion coefficients

$$C_k \equiv \frac{1}{k!} \left. \frac{d^k C_{\text{corr}}(t)}{dt^k} \right|_{t \rightarrow 0^+} \quad (37)$$

are finite real numbers with

$$C_{2k-1} = 0, \text{ for } k < K, \quad (38a)$$

$$C_{2K-1} \neq 0. \quad (38b)$$

The leading odd-order term in the short-time expansion of $C_{\text{corr}}(t)$ is $C_{2K-1}|t|^{2K-1}$. An example is $C_{\text{corr}}(t) = e^{-|t|^3}$ with

$C_1 = 0$ and $C_3 = -1$. We assume that the noise correlation decreases smoothly at long correlation times, that is,

$$\lim_{t \rightarrow \infty} \frac{d^k}{dt^k} C_{\text{corr}}(t) = 0, \text{ for } k = 0, 1, \dots, \quad (39)$$

and

$$I_L(\omega) \equiv \int_0^{\infty} e^{i\omega t} \frac{d^L}{dt^L} C_{\text{corr}}(t) dt, \quad (40)$$

vanishes at large ω for $L = 0, 1, \dots$ [54]. The noise correlations that decay in the correlation time smoothly without fast oscillation satisfy Eq. (40). For example, the noise correlation $e^{-|t|}$ has $I_L(\omega) = i(-1)^L/(\omega + i) \rightarrow 0$ when $\omega \rightarrow \infty$.

We consider the high frequency behavior of the Fourier transform of $C_{\text{corr}}(t)$,

$$S_{2K}(\omega) = \int_{-\infty}^{\infty} C_{\text{corr}}(t) e^{i\omega t} dt. \quad (41)$$

Integration by parts $L \geq (2K+1)$ times gives

$$S_{2K}(\omega) = 2\Re \left[\sum_{r=1}^L \frac{(r-1)!}{(-i\omega)^r} C_{r-1} + \frac{I_L(\omega)}{(-i\omega)^L} \right], \quad (42)$$

where we have used Eqs. (37) and (39).

Using Eqs. (38) and (40), we obtain for large ω ,

$$S_{2K}(\omega) \approx 2 \frac{(2K-1)!}{(i\omega)^{2K}} C_{2K-1} + O\left(\frac{1}{\omega^{2K+1}}\right), \quad (43)$$

which is a power-law decrease at high frequencies.

When the noise correlation expansion contains only even-order terms, from Eq. (42) we have the noise spectrum $S_{\text{even}}(\omega) = 2\Re I_L(\omega)/(-i\omega)^L$ for an arbitrarily large L . From the assumption $\lim_{\omega \rightarrow \infty} I_L(\omega) = 0$, we have $\lim_{\omega \rightarrow \infty} S_{\text{even}}(\omega)\omega^L = 0$ for an arbitrarily large L and therefore the noise spectrum has a hard high-frequency cutoff. One example is the correlation function e^{-t^2} , which has the noise spectrum of exponential form $\sim \exp(-\frac{\omega^2}{4})$, and obviously the UDD sequence can suppress the noise effect order by order. The large ω behavior of other correlation functions of the form $\exp(-\sum_{j=1}^P \alpha_{2j} t^{2j})$ can be calculated by the saddle point integration method, which gives a result of an exponential decrease at high frequencies (i.e., hard cutoff). For example, when ω is very large, $\int_{-\infty}^{\infty} e^{-t^4} e^{i\omega t} dt \simeq \frac{1}{2} \Im \sqrt{\frac{2\pi}{a(\omega)}} e^{g(\omega)}$, where $g(\omega) = 3(\frac{\omega}{4})^{\frac{4}{3}} e^{-i2\pi/3}$ and $a(\omega) = 12(\frac{\omega}{4})^{2/3} e^{i2\pi/3}$.

We now consider the noise spectrum

$$S_P(\omega) \approx \frac{1}{\omega^P}, \text{ for } \omega \geq \Omega, \quad (44)$$

approximated by a power law ($P > 0$) decrease beyond the frequency Ω . The noise correlation function is

$$\overline{\beta(t)\beta(0)} = C_{\Omega, \text{cutoff}}(t) + C_{\Omega, P}(t), \quad (45)$$

where the correlation due to high-frequency noise

$$C_{\Omega, P}(t) = 2\Re \int_{\Omega}^{\infty} \frac{d\omega}{2\pi} \frac{1}{\omega^P} e^{-i\omega t}, \quad (46)$$

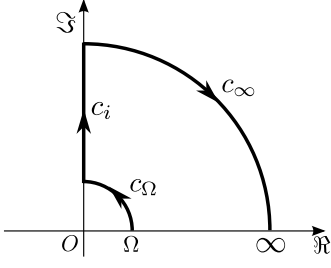


FIG. 1: The paths for the integral of $1/z^P$.

and the noise correlation at low frequencies

$$C_{\Omega, \text{cutoff}}(t) = 2\Re \int_{c_\Omega}^{\Omega} \frac{d\omega}{2\pi} S_P(\omega) e^{-i\omega t}, \quad (47)$$

causes additional decoherence. The effect from $C_{\Omega, \text{cutoff}}(t)$ can be eliminated efficiently by DD sequences, such as UDD sequences, as the noise spectrum of $C_{\Omega, \text{cutoff}}(t)$ has a hard cutoff Ω .

As $C_{\Omega, P}(t)$ is an even function, we just calculate the integral for the case of $t > 0$. We have (for $t > 0$)

$$C_{\Omega, P}(t) = \frac{\Re}{\pi} \left[\int_{c_\Omega} + \int_{c_i} + \int_{c_\infty} \right] \frac{1}{z^P} e^{izt} dz,$$

where the paths c_Ω , c_i , and c_∞ are shown in Fig. (1).

Since the maximum of $1/z^P \rightarrow 0$ as $|z| \rightarrow \infty$ in the upper half-plane, the contribution $\int_{c_\infty} \frac{1}{z^P} e^{izt} dz = 0$. The contribution

$$\frac{\Re}{\pi} \int_{c_i} \frac{1}{z^P} e^{izt} dz = \frac{\Re}{\pi} \int_{\Omega}^{\infty} i^{1-P} y^{-P} e^{-yt} dy \quad (48)$$

vanishes for even P . And

$$\begin{aligned} \frac{\Re}{\pi} \int_{c_\Omega} \frac{1}{z^P} e^{izt} dz &= \frac{\Re}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{\Omega^P e^{iP\theta}} e^{i\Omega t e^{i\theta}} (i\Omega) e^{i\theta} d\theta \\ &= I_{c_\Omega}^{(1)} + I_{c_\Omega}^{(2)}, \end{aligned} \quad (49)$$

where

$$I_{c_\Omega}^{(1)} = \frac{\Re}{\pi} \sum_{r=0, r \neq P-1}^{\infty} \frac{1}{r!} \frac{\Omega^{1-P} (i\Omega t)^r}{(r+1-P)} (i^{r+1-P} - 1), \quad (50a)$$

$$I_{c_\Omega}^{(2)} = \frac{1}{2} \Re \frac{i^P t^{P-1}}{(P-1)!} \sum_{r=0}^{\infty} \delta_{r, P-1}. \quad (50b)$$

For even P , $I_{c_\Omega}^{(1)}$ is an expansion of even powers of t , and

$$I_{c_\Omega}^{(2)} = \frac{(-1)^{P/2}}{2(P-1)!} |t|^{P-1}, \text{ for even } P, \quad (51)$$

is the leading odd order expansion term of the noise correlation function.

As an example, we consider the flowing noise spectrum,

$$S'_{2K}(\omega) \equiv \frac{1}{1 + \omega^{2K}}, \quad (52)$$

for $K \in \{1, 2, \dots\}$, which can be transformed to a more general form of noise $\frac{c}{a + (\omega/\omega_d)^{2K}}$ by using some scaling of the amplitude and ω . For example, the measured ambient noise for ions in a Penning trap has an approximate $1/\omega^4$ spectrum at high frequencies and a flat dependence at low frequencies [33], which approximately corresponds to the noise spectrum Eq. (52) with $K = 2$. The corresponding correlation function of Eq. (52) is obtained by the inverse transform

$$\overline{\beta(t)\beta(0)}_{2K} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{1 + \omega^{2K}}. \quad (53)$$

Using the residue theorem, we have

$$\overline{\beta(t)\beta(0)}_{2K} = \frac{i}{2K} \sum_{n=0}^{K-1} \frac{\exp[-ie^{i\frac{\pi}{2K}(2n+1)}|t|]}{\exp[i\frac{\pi}{2K}(2n+1)(2K-1)]}. \quad (54)$$

Expanding the right-hand side in powers of t , we get

$$\overline{\beta(t)\beta(0)}_{2K} = \frac{i}{2K} \sum_{m=0}^{\infty} \frac{(-i|t|)^m}{m!} \sum_{n=0}^{K-1} e^{i\frac{\pi}{2K}(2n+1)(m-2K+1)}, \quad (55)$$

and summing the terms gives

$$\overline{\beta(t)\beta(0)}_{2K} = \frac{-i}{2K} \sum_{m=0}^{\infty} \frac{(-i|t|)^m}{m!} \frac{e^{i\frac{(m-2K+1)\pi}{2K}} [(-1)^m + 1]}{e^{i(m+1)\pi/K} - 1}. \quad (56)$$

In Eq. (56), the leading odd-order term of the time expansion is $\frac{(-1)^K}{2(2K-1)!} |t|^{2K-1}$, as predicted by Eq. (51). By Theorem 1, we have the following corollary.

Corollary 1. *Dynamical decoupling can only suppress the decoherence up to an error $O(T^{2K+1})$ in the time-expansion of $\chi(T)$ for the noise spectrum with high-frequency asymptote $\sim \omega^{-2K}$.*

For a more general noise spectrum with high-frequency asymptote $\omega^{-\alpha}$, one can not suppress the error higher than the order of $O(T^{m_\alpha+1})$ with m_α being the smallest even number larger than α . This is because comparing with ω^{-m_α} , the asymptotic noise spectrum $\frac{1}{1+\omega^\alpha}$ with $\alpha < m_\alpha$ decreases slower and the overlap between $S(\omega)$ and $|\tilde{F}(\omega)|^2$ is larger.

The $1/f$ noise is a special case (the case of $P = 1$) to be discussed. The noise correlation function $C_{\Omega, P=1}(t) = \frac{1}{\pi} \int_{\Omega t}^{\infty} \cos z/z dz$ diverges at $t \rightarrow 0$; however, the decoherence function $\chi_{P=1}(T)$ [Eq. (11)] induced by the high-frequency noise is finite. When there is no DD control, i.e., for the case of free induction decay,

$$\begin{aligned} \chi_{P=1}(T) &= \frac{1}{2\pi\Omega^2} [1 - \cos(\Omega T) + \Omega T \sin(\Omega T)] \\ &\quad + \frac{T^2}{2\pi} \int_{\Omega T}^{\infty} \frac{\cos z}{z} dz, \end{aligned} \quad (57)$$

which has the short-time expansion

$$\chi_{P=1}(T) = \frac{1}{4\pi} (-2 \ln(\Omega T) + 3 - 2\gamma_e) T^2 + O(\Omega^2 T^4), \quad (58)$$

with the Euler constant $\gamma_e \approx 0.577$. Notice the interesting result that $\chi_{P=1}(T)$ does not have the simple power law scaling at short time T . After eliminating the effect of low-frequency noise by high-order DD sequences which satisfy $\{\Lambda_m = 0\}_{m=1}^M$ (Eq. (31)) with $2M > P - 1$, $\chi_P(T)$ has a power law scaling $\chi_P(T) = O(T^{P+1})$ [46], and we have $\chi_{P=1}(T) = O(T^2)$ when $\Lambda_1 = 0$.

III. SEQUENCE OPTIMIZATION IN SHORT-TIME LIMIT

The aim of optimal DD is to minimize $\chi(T)$. As indicated in Eq. (35), a smaller $\tilde{F}(\omega)$ at low frequencies will give a smaller ϕ_{2M-1} . Here we focus on DD with ideal instantaneous π pulses. We use more pulses to construct a more efficient modulation function $F(t) = F_\pi(t)$ to minimize ϕ_{2M-1} , with the conditions $\{\phi_{2m} = 0\}_{m=0}^{M-1}$ [Eq. (31)]. Using the method of Lagrange multipliers, we need to solve a set of nonlinear equations as

$$\nabla_{\{y,T\}} G_M = 0, \quad (59a)$$

$$G_M \equiv \sum_{j=1}^M y_j \Lambda_j^{(\pi)} + \phi_{2M-1}, \quad (59b)$$

$$\nabla_{\{y,T\}} \equiv \left(\frac{\partial}{\partial T_1}, \dots, \frac{\partial}{\partial T_N}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_M} \right). \quad (59c)$$

The introduced variables $\{y_j\}$ are the Lagrange multipliers. Note that the sequence optimization in Ref. [46] also used the constraints $\{\Lambda_m^{(\pi)} = 0\}$, but the constraints were used there to guarantee the convergence of the calculation of $\chi(T)$. Here, the constraints eliminate the lowest orders of errors ($\{\phi_{2m} = 0\}_{m=0}^{M-1}$) [see Eq. (31)] in short-time limit. In particular, the decoherence from inhomogeneous broadening is eliminated when $\phi_0 = 0$.

We calculate Eq. (16) by separating the domain of integration according to the value of $F_\pi(t_1)F_\pi(t_2)$. For $k \geq 0$, we obtain

$$\begin{aligned} \phi_k = & \frac{-1}{(k+1)(k+2)} \left[4 \sum_{j=2}^N \sum_{i=1}^{j-1} (T_j - T_i)^{k+2} (-1)^{i+j} \right. \\ & + (T_{N+1} - T_0)^{k+2} (-1)^{N+1} + 2 \sum_{j=1}^N (T_j - T_0)^{k+2} (-1)^j \\ & \left. + 2 \sum_{i=1}^N (T_{N+1} - T_i)^{k+2} (-1)^{N+1+i} \right]. \end{aligned} \quad (60)$$

Then we have

$$\begin{aligned} \frac{\partial \phi_{2M-1}}{\partial T_k} = & \frac{(-1)^k}{M} \left[2 \sum_{j=k+1}^N (T_j - T_k)^{2M} (-1)^j - (T_k - T_0)^{2M} \right. \\ & \left. + (T_{N+1} - T_k)^{2M} (-1)^{N+1} - 2 \sum_{j=1}^{k-1} (T_k - T_j)^{2M} (-1)^j \right]. \end{aligned} \quad (61)$$

For the special case of $M = 1$, $G_1 = y_1 \Lambda_1^{(\pi)} + \phi_1$, we find that the CPMG sequences are solutions to Eq. (59). The timing of

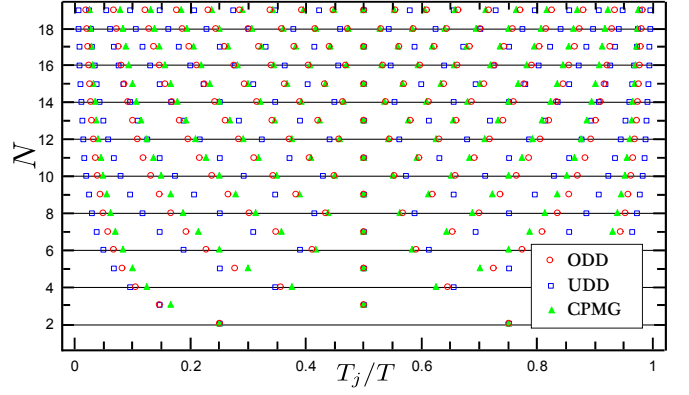


FIG. 2: (Color online). Comparison of the ODD, UDD and CPMG sequences for different pulse number N . Squares (blue), triangles (green), and circles (red) correspond to UDD, CPMG, and ODD. The ODD sequences are optimized to minimize ϕ_3 under the constraints $\phi_0 = \phi_2 = 0$.

an N -pulse CPMG sequence reads

$$T_j^{\text{CPMG}} = \frac{2j-1}{2N} T, \text{ for } j = 1, \dots, N. \quad (62)$$

The CPMG sequences obviously satisfy the constraint $\Lambda_1^{(\pi)} = 0$ [see Eq. (20)], which is the so-called echo condition that eliminates the effect of static inhomogeneous broadening. Eqs. (61) and (20) give

$$\left. \frac{\partial \phi_1}{\partial T_k} \right|_{\text{CPMG}} = (-1)^{k+1} \frac{T^2}{4N^2} [1 + (-1)^N], \quad (63a)$$

$$\left. y_1 \frac{\partial \Lambda_1^{(\pi)}}{\partial T_k} \right|_{\text{CPMG}} = 2(-1)^{k+1} y_1. \quad (63b)$$

Thus the CPMG sequences also satisfy Eq. (59) with $y_1 = -\frac{T^2}{8N^2} [1 + (-1)^N]$, so they are at least the locally optimal pulse sequences. It has been proved that CPMG sequences are the most efficient pulse sequences in protecting the qubit coherence against telegraph-like noise in the short-time limit [47]. With numerical evidence, we conjecture that it is also true that the CPMG sequences are optimal in the short-time limit when the time expansion of the noise correlation function has the two leading terms C_0 and $C_1|t|$.

For other cases of minimizing ϕ_{2M-1} with the condition $\{\Lambda_m^{(\pi)} = 0\}_{m=1}^M$, one can see that the short-time optimized DD (ODD) coincides with UDD for pulse number $N \leq M$. And as N increases, the ODD sequences gradually approach the CPMG sequences. For example, ODD for the noise correlation

$$\overline{\beta(t)\beta(0)} = C_0 + C_2 t^2 + C_3 |t|^3 + O(t^4), \quad (64)$$

is shown in Fig. 2 in comparison with UDD and CPMG. The ODD sequences resemble the CPMG sequences when N is large.

We show in Fig. 3(a) the performance of three DD schemes against the noise described by Eq. (64). A comparison is also

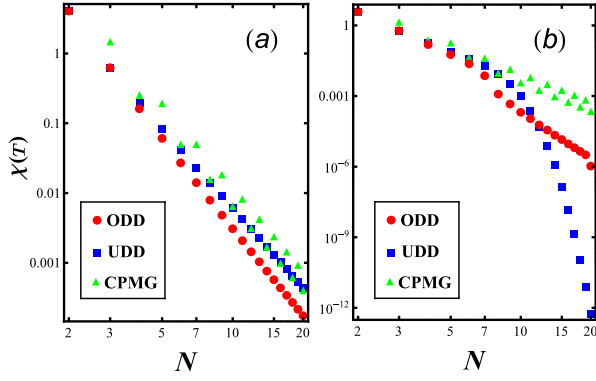


FIG. 3: (Color online). The decoherence function $\chi(T)$ as a function of pulse number N (a) without hard cutoff and (b) with a hard high-frequency cutoff $\omega_c = 40$. Here the noise spectrum $S(\omega) = \frac{\kappa}{1+\omega^4}$, with $\kappa = 10^5$ and $T = 0.5$. Squares (blue), triangles (green), and circles (red) correspond to UDD, CPMG, and ODD. The ODD sequences are the same as those shown in Fig. 2.

shown in Fig. 3(b) by considering a hard high-frequency cutoff $\omega_c = 40$. In Fig. 3(a), we can see that ODD sequences give better performance than UDD and CPMG sequences. These ODD sequences are optimal for a wide range of noise which has the noise correlation given by Eq. (64). When we introduce a high-frequency cutoff in the noise spectrum, as the case in Fig. 3(b), the ODD is the best initially when the number of pulses $N \lesssim \omega_c T/2 \approx 10$, and the UDD sequences become better and suppress the decoherence order by order when N is large and the hard cutoff is reached. In Fig. 3(b), for large N UDD is better than ODD, since the ODD sequences are optimized for soft-cutoff noise rather than hard-cutoff noise. In Fig. 3(a) the decreasing of $\chi(T)$ is a linear decrease in the double-logarithm plot, but in Fig. 3(b) it is much faster. This confirms that DD is not so efficient against soft-cutoff noise.

IV. SUMMARY AND CONCLUSIONS

We have studied the dynamical decoupling control of decoherence caused by Gaussian noise with soft cutoff. We have proved Theorem 1 which shows that, when the short-time expansion of noise correlation has the $(2K-1)$ th odd expansion term, DD can only suppress decoherence to $O(T^{2K+1})$. The existence of odd-order terms in the short-time expansion corresponds to a soft high-frequency cutoff in the noise spectrum (i.e., a power-law asymptote at high frequencies). For these noise spectra, we have derived the equations for pulse sequence optimization, which minimizes the leading odd-order decoherence function and eliminates even-order decoherence functions of lower orders. The ODD sequences obtained by this method coincide with the UDD sequences when the pulse number N is small, and they resemble CPMG sequences when N is large. For the special case that the short-time correlation function expansion has a linear term in time, the ODD sequences are exactly the CPMG sequences.

Although we derived Theorem 1 from a pure dephasing

model, we expect that the result of the existence of the largest decoupling order in short-time limit can be extended to the general decoherence model (including both dephasing and relaxation) of quantum systems. It is desirable to study the DD in suppressing the general decoherence of quantum systems in the soft-cutoff noise in the future.

Acknowledgments

We thank Yi-Fan Luo and Bobo Wei for discussions. This work was supported by the Hong Kong GRF CUHK402209, the CUHK Focused Investments Scheme, and the National Natural Science Foundation of China Project No. 11028510.

Appendix A: Proof of Equation (26)

To prove Eq. (26), we just need to prove

$$\lim_{R \rightarrow \infty} \int_{-R}^R dx \sum_{n=r}^{\infty} \frac{(\pm ix)^{n-r}}{n!} = \frac{\pi}{(r-1)!}, \text{ for } r \geq 1, \quad (\text{A1})$$

where the bounds in the integral guarantee that the modulation function $F(t)$ is a real function. Using

$$\int_{-R}^R dx \sum_{n=r}^{\infty} \frac{(\pm ix)^{n-r}}{n!} = \sum_{n=1}^{\infty} \frac{R^n}{(n+r-1)!n} (i^{n-1} + \text{c.c.}), \quad (\text{A2a})$$

$$\frac{1}{(r-1)!} \int_{-R}^R \frac{\sin x}{x} dx = \sum_{n=1}^{\infty} \frac{R^n}{n!n} \frac{1}{(r-1)!} (i^{n-1} + \text{c.c.}), \quad (\text{A2b})$$

and $\lim_{R \rightarrow \infty} \int_{-R}^R \frac{\sin x}{x} dx = \pi$, we just need to prove

$$\lim_{R \rightarrow \infty} \sum_{n=1}^{\infty} \left[\frac{R^n}{(n+r-1)!n} - \frac{R^n}{n!n(r-1)!} \right] (i^{n-1} + \text{c.c.}) = 0. \quad (\text{A3})$$

For $r = 1$, it obviously holds. For $r \geq 2$, we can show the difference

$$\Delta \equiv \sum_{n=1}^{\infty} \frac{R^n (i^{n-1} + \text{c.c.})}{(n+r-1)!n} \left[(r-1)! - \prod_{j=1}^{r-1} (n+j) \right] = O\left(\frac{1}{R}\right), \quad (\text{A4})$$

so $\lim_{R \rightarrow \infty} \Delta = 0$. By expanding the terms in the square brackets of Eq. (A4), we have

$$\Delta = \sum_{n=1}^{\infty} \frac{R^n (i^{n-1} + \text{c.c.})}{(n+r-1)!} \left(\sum_{k=0}^{r-2} a_k n^k \right), \quad (\text{A5})$$

where a_k is a number independent of n . We arrange the terms in the square brackets and get

$$\Delta = \sum_{n=1}^{\infty} \frac{R^n (i^{n-1} + \text{c.c.})}{(n+r-1)!} \left[\sum_{k=1}^{r-2} b_k \prod_{j=1}^k [(n+r-j)] + b_0 \right], \quad (\text{A6})$$

with b_j independent of n . After some simplification it becomes for $r \geq 2$

$$\sum_{k=1}^{r-1} \frac{b_{r-k-1}}{i^{k+1} R^k} \left(e^{iR} - \sum_{n=0}^{k-1} \frac{R^n}{n!} i^n \right) + \text{c.c.} = O\left(\frac{1}{R}\right). \quad (\text{A7})$$

Hence $\Delta = O(\frac{1}{R})$, and Eq. (26) is proved.

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- [1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University, Cambridge, 2000).
- [2] L.-M. Duan and G.-C. Guo, Phys. Rev. Lett. **79**, 1953 (1997).
- [3] P. Zanardi, Phys. Rev. A **57**, 3276 (1998).
- [4] D. A. Lidar, I. L. Chuang, and K. B. Whaley, Phys. Rev. Lett. **81**, 2594 (1998).
- [5] P. W. Shor, Phys. Rev. A **52**, 2493 (1995).
- [6] A. M. Steane, Proc. R. Soc. Lond. A **452**, 2551 (1996).
- [7] P. Zanardi, Phys. Lett. A **258**, 77 (1999).
- [8] L. Viola, E. Knill, and S. Lloyd, Phys. Rev. Lett. **82**, 2417 (1999).
- [9] W. Yang, Z.-Y. Wang, and R.-B. Liu, Front. Phys. **6**, 2 (2011).
- [10] E. L. Hahn, Phys. Rev. **80**, 580 (1950).
- [11] H. Y. Carr and E. M. Purcell, Phys. Rev. **94**, 630 (1954).
- [12] S. Meiboom and D. Gill, Rev. Sci. Instrum. **29**, 688 (1958).
- [13] M. Mehring, *Principles of High Resolution NMR in Solids* (Springer-Verlag, Berlin, 1983), 2nd ed.
- [14] K. Khodjasteh and D. A. Lidar, Phys. Rev. Lett. **95**, 180501 (2005).
- [15] K. Khodjasteh and D. A. Lidar, Phys. Rev. A **75**, 062310 (2007).
- [16] W. Yao, R.-B. Liu, and L. J. Sham, Phys. Rev. Lett. **98**, 077602 (2007).
- [17] W. M. Witzel and S. Das Sarma, Phys. Rev. B **76**, 241303(R) (2007).
- [18] W. X. Zhang, V. V. Dobrovitski, L. F. Santos, L. Viola, and B. N. Harmon, Phys. Rev. B **75**, 201302(R) (2007).
- [19] X. Peng, D. Suter, and D. A. Lidar, J. Phys. B: At. Mol. Opt. Phys. **44**, 154003 (2011).
- [20] G. A. Álvarez, A. Ajoy, X. Peng, and D. Suter, Phys. Rev. A **82**, 042306 (2010).
- [21] A. M. Tyryshkin, Z.-H. Wang, W. Zhang, E. E. Haller, J. W. Ager, V. V. Dobrovitski, and S. A. Lyon, arXiv:1011.1903v2.
- [22] Z.-H. Wang, W. Zhang, A. M. Tyryshkin, S. A. Lyon, J. W. Ager, E. E. Haller, and V. V. Dobrovitski, Phys. Rev. B **85**, 085206 (2012).
- [23] C. Barthel, J. Medford, C. M. Marcus, M. P. Hanson, and A. C. Gossard, Phys. Rev. Lett. **105**, 266808 (2010).
- [24] L. F. Santos and L. Viola, New J. Phys. **10**, 083009 (2008).
- [25] Z.-Y. Wang and R.-B. Liu, Phys. Rev. A **83**, 022306 (2011).
- [26] G. S. Uhrig, Phys. Rev. Lett. **98**, 100504 (2007), *ibid*, **106**, 129901 (2011).
- [27] B. Lee, W. M. Witzel, and S. Das Sarma, Phys. Rev. Lett. **100**, 160505 (2008).
- [28] G. S. Uhrig, New J. Phys. **10**, 083024 (2008).
- [29] W. Yang and R.-B. Liu, Phys. Rev. Lett. **101**, 180403 (2008).
- [30] G. S. Uhrig and D. A. Lidar, Phys. Rev. A **82**, 012301 (2010).
- [31] S. Pasini, T. Fischer, P. Karbach, and G. S. Uhrig, Phys. Rev. A **77**, 032315 (2008).
- [32] B. Fauseweh, S. Pasini, and G. S. Uhrig, Phys. Rev. A **85**, 022310 (2012).
- [33] M. J. Biercuk, H. Uys, A. P. VanDevender, N. Shiga, W. M. Itano, and J. J. Bollinger, Nature **458**, 996 (2009).
- [34] J. Du, X. Rong, N. Zhao, Y. Wang, J. Yang, and R.-B. Liu, Nature **461**, 1265 (2009).
- [35] M. J. Biercuk, H. Uys, A. P. VanDevender, N. Shiga, W. M. Itano, and J. J. Bollinger, Phys. Rev. A **79**, 062324 (2009).
- [36] H. Uys, M. J. Biercuk, and J. J. Bollinger, Phys. Rev. Lett. **103**, 040501 (2009).
- [37] E. R. Jenista, A. M. Stokes, R. T. Branca, and W. S. Warren, J. Chem. Phys. **131**, 204510 (2009).
- [38] G. S. Uhrig, Phys. Rev. Lett. **102**, 120502 (2009).
- [39] J. R. West, B. H. Fong, and D. A. Lidar, Phys. Rev. Lett. **104**, 130501 (2010).
- [40] G. Quiroz and D. A. Lidar, Phys. Rev. A **84**, 042328 (2011).
- [41] W.-J. Kuo and D. A. Lidar, Phys. Rev. A **84**, 042329 (2011).
- [42] L. Jiang and A. Imambekov, Phys. Rev. A **84**, 060302 (2011).
- [43] M. Mukhtar, T. B. Saw, W. T. Soh, and J. Gong, Phys. Rev. A **81**, 012331 (2010).
- [44] Z.-Y. Wang and R.-B. Liu, Phys. Rev. A **83**, 062313 (2011).
- [45] Ł. Cywiński, R. M. Lutchyn, C. P. Nave, and S. Das Sarma, Phys. Rev. B **77**, 174509 (2008).
- [46] S. Pasini and G. S. Uhrig, Phys. Rev. A **81**, 012309 (2010).
- [47] K. Chen and R.-B. Liu, Phys. Rev. A **82**, 052324 (2010).
- [48] K. Khodjasteh, T. Erdélyi, and L. Viola, Phys. Rev. A **83**, 020305 (2011).
- [49] G. S. Uhrig and S. Pasini, New J. Phys. **12**, 045001 (2010).
- [50] Z.-Y. Wang and R.-B. Liu, unpublished.
- [51] P. W. Anderson, J. Phys. Soc. Jpn. **9**, 316 (1954).
- [52] R. Kubo, J. Phys. Soc. Jpn. **9**, 935 (1954).
- [53] P. R. Berman and R. G. Brewer, Phys. Rev. A **32**, 2784 (1985).
- [54] C. P. Sun and X. J. Liu, Acta Phys. Sin. **5**, 343 (1996).